Church-Rosser Theorem and Compositional
Z-Property

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The Church-Rosser theorem is one of the most fundamental properties on rewriting systems. In order to
prove the theorem for beta-equality, Church and Rosser extracted the key property called confluence or
so-called Strip lemma. Then the theorem can be proved by induction on the number of peaks from the
key property. Here, the key property can be verified by using well-known notions such as parallel reduc-
tion and residuals. Although confluence and the Church-Rosser property are equivalent to each other, the
property of confluence is a special case of the theorem. First, we investigate directly
the theorem from the
viewpoint of Takahashi translation, which provides a new and constructive proof of the theorem. The proof
method has recently been established by the first author. Next, we show that the method is available as
well under a general framework of the compositional Z (Nakazawa-Fujita) that makes it possible to apply a
divide-and-conquer method for proving the Church-Rosser property.

1 Introduction

The Church-Rosser theorem [3] is one of the most fundamental properties on rewriting systems, which
guarantees uniqueness of computation and consistency of a formal system. For instance, for
proof trees and formulae of logic the unique normal forms of the corresponding terms and types in a
Pure Type System (PTS) can be chosen as their denotations [24] via the Curry-Howard isomorphism.

The Church-Rosser theorem for beta-equality states that if \( M =_\beta N \) then there exists \( P \) such that
\( M \to P \) and \( N \to P \). Here, we write \( M =_\beta N \) iff \( M \) is obtained from \( N \) by a finite series of reductions
\((\to)\) and reversed reductions \((\leftarrow)\). As the Church-
Rosser theorem for beta-reduction (confluence) has
been well studied, to the best of our knowledge the
Church-Rosser theorem for beta-equality is always secondary proved as a corollary from the theorem for

In order to prove the theorem, Church and Rosser
extracted the key property of confluence. The prop-
erty states that if \( M \to N_1 \) and \( M \to N_2 \) then we
have \( N_1 \to P \) and \( N_2 \to P \) for some \( P \). Two proof

techniques of the property are well known; trac-
ing the residuals of redexes along a sequence of re-
ductions [3] [2] [9], and working with parallel reduc-
tion [4] [2] [9] [22] known as the method of Tait and
Martin-Lof. Moreover, a simpler proof of the theo-
rem is established only with Takahashi’s translation
[22] (the Gross-Knuth reduction strategy [2]), but
with no use of parallel reduction [14] [5].

One of our motivation is to analyze quantitative
properties in general of reduction systems. For in-
stance, measures for developments are investigated
by Hindley [8] and de Vrijer [21]. Statman [19]
proved that deciding the \( \beta\eta \)-equality of typable \( \lambda \)-
terms is not elementary recursive. Schwichtenberg
[17] analysed the complexity of normalization in

\( \text{Church-Rosser theorem} \)
the simply typed lambda-calculus, and showed that
the number of reduction steps necessary to reach
the normal form is bounded by a function at the
fourth level of the Grzegorczyk hierarchy \( \varepsilon^4 \) [7],
i.e., a non-elementary recursive function. Ketema
and Simonsen [11] extensively studied valley sizes of
confluence and the Church-Rosser property in term
rewriting and \( \lambda \)-calculus as a function of given term
sizes and reduction lengths. However, there are no
known bounds for the Church-Rosser theorem for
\( \beta \)-equality up to our knowledge.

In this study, we are also interested in quantita-
tive analysis of the witness of the Church-Rosser
theorem: how to find common contractums with
the least size and with the least number of reduc-
tion steps. For the theorem for \( \beta \)-equality \( (M =_\beta N \implies M \rightarrow^{l_3} P \text{ and } N \rightarrow^{l_4} P \text{ for some } P) \),
we study functions that set bounds on the least size
of a common contractum \( P \), and the least number
of reduction steps \( l_3 \) and \( l_4 \) required to arrive at
a common contractum, involving the term sizes of
\( M \) and \( N \), and the length of \( \equiv_{\beta} \). For the theo-
rem for \( \beta \)-reduction \( (M \rightarrow^{l_1} N_1 \text{ and } M \rightarrow^{l_2} N_2 \implies N_1 \rightarrow^{l_3} P \text{ and } N_2 \rightarrow^{l_4} P \text{ for some } P) \),
we study functions that set bounds on the least size
of a common contractum \( P \), and the least number
of reduction steps \( l_3 \) and \( l_4 \) required to arrive at
a common contractum, involving the term size of \( M \)
and the lengths of \( l_1 \) and \( l_2 \).

In this paper, first we investigate directly the
Church-Rosser theorem for \( \beta \)-equality con-
structively from the viewpoint of Takahashi translation
[22]. Although the two statements are equivalent to
each other, the theorem for \( \beta \)-reduction is a special
case of that for \( \beta \)-equality. Our investigation shows
that a common contractum of \( M \) and \( N \) such that
\( M =_\beta N \) is determined by (i) \( M \) and the number
of occurrences of reduction \( (\rightarrow) \) appeared in \( =_\beta \),
and also by (ii) \( N \) and that of reversed reduction
\( (\leftarrow) \). The main lemma plays a key role and reveals
a new invariant involved in the equality \( =_\beta \), inde-
dependently of an exponential combination of reduc-
tion and reversed reduction. In terms of iteration of
Takahashi translation, this characterization of the
Church-Rosser theorem makes it possible to anal-
yse how large common contractums are and how
many reduction-steps are required to obtain them.
From this, we obtain an upper bound function for
the theorem in the fourth level of the Grzegorczyk
hierarchy.

Next, we demonstrate that the proof method
is available as well under a general framework of
the compositional \( Z \) [15]. The compositional \( Z \-
property is an extension of the so-called \( Z \)-property
[5], which makes it possible to apply a divide-and-
conquer method for proving confluence. For this
extension, the measure functions constructed for
quantitative analysis of the Church-Rosser theorem
are abstracted as fundamental modules of bound
functions. The paper makes a contribution to quan-
titative analysis of abstract rewriting systems un-
der the framework of the compositional \( Z \).

This paper is organized as follows. Section 1 is
devoted to background, related work, and our con-
tribution of this paper. Section 2 gives prelimi-
naries including basic definitions and notions. Fol-
lowing [6], Section 3 provides the new proof of the
Church-Rosser theorem for \( \beta \)-equality. Based on
this, from the viewpoint of abstract rewriting sys-
tems, reduction length for the theorem is analyzed
in Section 4. Section 5 recalls the compositional \( Z \-
property [15]. Section 6 demonstrates quantitative
analysis of reduction systems under the framework
of the compositional \( Z \), and this part is a new result
of the paper. Section 7 concludes with remarks, re-
lated work, and further work.

The paper is an extended abstract, and see [6] for
the details of the new proof of the Church-Rosser
theorem and quantitative analysis of the witness,
and see also [15] for the details of the compositional
2 Preliminaries

The set of $\lambda$-terms denoted by $\Lambda$ is defined with a countable set of variables as follows.

Definition 2.1 ($\lambda$-terms).

$$M, N, P, Q \in \Lambda ::= x \mid (\lambda x. M) \mid (MN)$$

We write $M \equiv N$ for the syntactical identity under renaming of bound variables. We suppose that every bound variable is distinct from free variables. The set of free variables in $M$ is denoted by $FV(M)$.

If $M$ is a subterm of $N$ then we write $M \sqsubseteq N$ for this.

Definition 2.2 ($\beta$-reduction). One step $\beta$-reduction $\rightarrow$ is defined as follows, where $M[x := N]$ denotes a result of substituting $N$ for every free occurrence of $x$ in $M$.

1. $(\lambda x. M)N \rightarrow M[x := N]$
2. If $M \rightarrow N$ then $PM \rightarrow PN$, $MP \rightarrow MP$, and $\lambda x. M \rightarrow \lambda x. N$.

A term in the form of $\lambda x. PQ \sqsubseteq M$ is called a redex of $M$. A redex is denoted by $R$ or $S$, and we write $R : M \rightarrow N$ if $N$ is obtained from $M$ by contracting the redex $R \sqsubseteq M$. We write $\rightarrow$ for the reflexive and transitive closure of $\rightarrow$. If $R_1 : M_0 \rightarrow M_1, \ldots, R_n : M_{n-1} \rightarrow M_n (n \geq 0)$, then for this we write $R_0 \ldots R_n : M_0 \rightarrow M_n$, and the reduction sequence is denoted by the list $[M_0, M_1, \ldots, M_n]$. For operating on a list, we suppose fundamental list functions, append, reverse, tail (cdr), map and max.

Definition 2.3 ($\beta$-equality). A term $M$ is $\beta$-equal to $N$ with reduction sequence $ls$, denoted by $M =_{\beta} N$ with $ls$ is defined as follows:

1. If $M \rightarrow N$ with reduction sequence $ls$, then $M =_{\beta} N$ with $ls$.
2. If $M =_{\beta} N$ with $ls$, then $N =_{\beta} M$ with reverse($ls$).
3. If $M =_{\beta} P$ with $ls_1$ and $P =_{\beta} N$ with $ls_2$, then $M =_{\beta} N$ with append($ls_1, \text{tail}(ls_2)$).

Note that $M =_{\beta} N$ with reduction sequence $ls$ iff there exist terms $M_0, \ldots, M_n (n \geq 0)$ in this order such that $ls = [M_0, \ldots, M_n]$, $M_0 \equiv M, M_n \equiv N$, and either $M_i \rightarrow M_{i+1}$ or $M_{i+1} \rightarrow M_i$ for each $0 \leq i < n-1$. In this case, we say that the length of $=_{\beta}$ is $n$, denoted by $=_{\beta}^n$. The arrow in $M_i \rightarrow M_{i+1}$ is called a right arrow, and the arrow in $M_{i+1} \rightarrow M_i$ is called a left arrow, denoted also by $M_i \leftarrow M_{i+1}$.

Definition 2.4 (Term size). Define a function $| | : \Lambda \rightarrow \mathbb{N}$ as follows.

1. $|x| = 1$
2. $|\lambda x. M| = 1 + |M|$
3. $|MN| = 1 + |M| + |N|$

Definition 2.5 (Takahashi’s $*$ and iteration). The notion of Takahashi translation $M^* \ [22]$, that is, the Gross-Knuth reduction strategy $[2]$ is defined as follows.

1. $x^* = x$
2. $(\lambda x. M)^* = M^*[x := N^*]$
3. $(MN)^* = M^*N^*$
4. $(\lambda x. M)^* = \lambda x. M^*$

The 3rd case above is available provided that $M$ is not in the form of $\lambda$-abstraction. We write an iteration of the translation $[23]$ as follows.

1. $M^{0*} = M$
2. $M^{n*} = (M^{(n-1)*})^*$

We write $\sharp(x \in M)$ for the free occurrence number of the variable $x$ in $M$.

Lemma 2.6. $|M[x := N]| = |M| + \sharp(x \in M) \times (|N| - 1)$.

Proof. By straightforward induction on $M$.

Definition 2.7 (Redex($M$)). A set of all redex occurrences in a term $M$ is denoted by Redex($M$).

The cardinality of the set Redex($M$) is denoted by $\sharp$Redex($M$).

Lemma 2.8 ($\sharp$Redex($M$)). We have $\sharp$Redex($M$) \leq \frac{1}{2} |M| - 1$ for $|M| \geq 4$.

Proof. Note that $\sharp$Redex($M$) = 0 for $|M| < 4$. By
straightforward induction on $M$ for $|M| \geq 4$. □

Lemma 2.9 (Substitution). If $M_1 \rightarrow^l N_1$ and $M_2 \rightarrow^l N_2$, then $M_1[x := M_2] \rightarrow^l N_1[x := N_2]$ where $l = l_1 + l_2(x \in M_1) \times l_2$. 

Proof. By induction on the structure of $M_1 \rightarrow^l N_1$. The case of $l_1 = 0$ requires induction on $M_1 \equiv N_1$. □

Proposition 2.10 (Term size after n-step reduction). If $M \rightarrow^n N$ ($n \geq 1$) then $|N| < 8 \left(\frac{|M|}{8}\right)^n$. 

Proof. By straightforward induction on $n$. □

Lemma 2.11 (Size of $M^*$). We have $|M^*| \leq 2^{|M|-1}$. 

Proof. By straightforward induction on $M$. □

3 New proof of the Church-Rosser theorem for $\beta$-equality

Proposition 3.1 (Complete development). We have $M \rightarrow^l M^*$ where $l \leq \frac{1}{2}|M| - 1$ for $|M| \geq 4$.

Proof. By induction on the structure of $M$. Otherwise by the minimal complete development [9] with respect to $\text{Redex}(M)$, where $l \leq \frac{1}{2}\text{Redex}(M) \leq \frac{1}{2}|M| - 1$ from Lemma 2.8. □

Definition 3.2 (Iteration of exponentials $2^m_n$, $F(m, n)$). Let $m$ and $n$ be natural numbers.
1. (1) $2^m_0 = m$; (2) $2^{m+1}_n = 2^{m}_n$.
2. (1) $F(m, 0) = m$; (2) $F(m, n+1) = 2^{F(m, n)}$.

Proposition 3.3 (Length to $M^*$). If $M \rightarrow M^* \rightarrow \cdots \rightarrow M^{**}$, then the reduction length $l$ with $M \rightarrow^l M^{**}$ is bounded by $\text{Len}(|M|, n)$, such that

$$\text{Len}(|M|, n) = \begin{cases} 0 & \text{for } n = 0 \\ \frac{1}{2} \sum_{k=0}^{n-1} F(|M|, k) - n & \text{for } n \geq 1 \end{cases}$$

and hence $|M^{**}| \leq F(|M|, k) < 2^{|M|}_k$ for $k \geq 1$. Let $M \rightarrow^l M^* \rightarrow^l \cdots \rightarrow^l M^{**}$. Then from Proposition 3.1, each $l_k$ is bounded by $F(|M|, k - 1)$:

$$l_k \leq \frac{1}{2} |M^{(k-1)*}| - 1 \leq \frac{1}{2} F(|M|, k - 1) - 1$$

Therefore, $l$ is bounded by $\text{Len}(|M|, n)$ that is smaller than $2^{|M|-1}_{n-1}$ for $n \geq 1$.

$$l \leq \sum_{k=1}^{n-1} l_k$$

Lemma 3.4 (Cofinal property). If $M \rightarrow N$ then $N \rightarrow^l M^*$ where $l \leq \frac{1}{2}|M| - 1$ for $|M| \geq 4$.

Proof. By induction on the derivation of $M \rightarrow N$. □

Lemma 3.5. $M^*[x := N^*] \rightarrow^l (M[x := N])^*$ with $l \leq |M^*| - 1$.

Proof. By induction on the structure of $M$. □

Proposition 3.6 (Monotonicity).
1. If $M \rightarrow N$ then $M^* \rightarrow^l N^*$ with $l \leq |M^*| - 1$.
2. If $M \rightarrow^m N$, then $M^* \rightarrow^l N^*$ where $l \leq 2^{|M|} - 2^{m-1} - m$.

Proof. 1. By induction on the derivation of $M \rightarrow N$.
2. From Proposition 2.10, Proposition 3.6 (1) and Lemma 2.11. □

Lemma 3.7 (Main lemma [6]). Let $M = b^k$ with length $k = l + r$, where $r$ is the number of occurrences of right arrow $\rightarrow$ in $= b$, and $l$ is that of left arrow $\leftarrow$ in $= b$. Then we have both $M^* = N$ and
$M \rightarrow N^{s*}$.

Proof. By induction on the length of $=^k$.  
(1) Case of $k = 1$ is handled by Lemma 3.4.  
(2-1) Case of $(k+1)$, where $M = \delta^n M_k \rightarrow M_{k+1}$:  
From the induction hypothesis, we have $M_k \rightarrow M^{*}$ and $M \rightarrow M_k^{s*}$, where $l + r = k$.  
From $M_k \rightarrow M_{k+1}$, Lemma 3.4 gives $M_{k+1} \rightarrow M^{s*}_k$, and then $M_k^{s*} \rightarrow M^{(r+1)*}$ from the induction hypothesis $M_k \rightarrow M^{*}$ and Proposition 3.6. Hence, we have $M_{k+1} \rightarrow M^{(r+1)*}$. On the other hand, we have $M_k^{s*} \rightarrow M_{k+1}^{s*}$ from $M_k \rightarrow M_{k+1}$ and the repeated application of Proposition 3.6. Then the induction hypothesis $M \rightarrow M_k^{s*}$ derives $M \rightarrow M_k^{s*+1}$, where $l + (r + 1) = k + 1$.

(2-2) Case of $(k+1)$, where $M = \delta^n M_k \leftarrow M_{k+1}$:  
From the induction hypothesis, we have $M_k \rightarrow M^{*}$ and $M \leftarrow M_k^{s*}$, where $l + r = k$, and hence $M_{k+1} \rightarrow M^{*}$. From $M_{k+1} \rightarrow M_k$ and Lemma 3.4, we have $M_k \rightarrow M_{k+1}^{s*}$, and then $M_k^{s*} \rightarrow M_k^{(r+1)*}$. Hence, $M \leftarrow M_k^{(r+1)*}$ from the induction hypothesis $M \rightarrow M_k^{s*}$, where $(l + 1) + r = k + 1$.

Given $M_0 = \delta^n M_k$ with reduction sequence $[M_0, \ldots, M_k]$, then for natural numbers $i$ and $j$ with $0 \leq i \leq j \leq k$, we write $\sharp[i, j]$ for the number of occurrences of right arrow $\rightarrow$ appeared in $M_i = \delta^{(j-i)} M_j$, and $\sharp[i, j]$ for that of left arrow $\leftarrow$ in $M_i = \delta^{(j-i)} M_j$. In particular, we have $\sharp[0, k] + \sharp[0, k] = k$.

Corollary 3.8 (Main lemma refined [6]). Let $M_0 = \delta^n M_k$ with reduction sequence $[M_0, M_1, \ldots, M_k]$. Let $r = \sharp[0, k]$ and $l = \sharp[0, k]$. Then we have $M_0 \rightarrow M_0^{s*}$ and $M_k^{s*} \leftarrow M_k$, where $m_i = \sharp[0, l] \leq \min\{l, r\}$.

Proof. From the main lemma, we have two reduction paths such that $M_0 \rightarrow M_0^{s*}$ and $M_k^{s*} \leftarrow M_k$, where the paths have a crossed point that is the term $M^{s*}$ for some $n \leq k$ as follows: Let $M_l$ be $\sharp[0, r]$, then $\sharp[r, k] = (l - m_l)$ and $\sharp[r, k] = m_l$. Hence, from the main lemma, we have $M_0 \rightarrow M_0^{s*} \leftarrow M_k$ where $m_l \leq \min\{l, r\}$. Moreover, we have $M_r \rightarrow M_k^{(l-m_l)*}$ by the main lemma again, and then $M_0^{m_l*} \rightarrow M_k^{(l-m_l+1)*}$ from the repeated application of Proposition 3.6. Therefore, we indeed have $M_0 \rightarrow M_0^{m_l*} \rightarrow M_k^{s*}$. Similarly, we have $M_0^{m_l*} \rightarrow M_0^{m_l*} \leftarrow M_k$ as well.

Observe that a crossed point $M_0^{m_l*}$ in Corollary 3.8 gives a “good” common contractum such that the number $m_l$, i.e., iteration of the translation * is minimum. Consider two reduction paths: (i) a reduction path from $M_0^{m_l*}$ to $M_0^{s*}$, and (ii) a reduction path from $M^{m_l*}$ to $M_k^{s*}$, see the picture in the proof of Corollary 3.8. In general, the reduction paths (i) and (ii) form the boundary line between common contractums and non-common ones. Let $B$ be a term in the boundary (i) or (ii). Then any term $M$ such that $B \rightarrow M$ is a common contractum of $M_0$ and $M_k$. In this sense, the term $M_0^{m_l*}$ with $0 \leq m_l \leq \min\{l, r\}$ can be considered as an optimum common reduct of $M_0$ and $M_k$ in terms of Takahashi translation. Moreover, the refined lemma gives a divide and conquer method such that $M_0 = \delta^n M_k$ is divided into $M_0 = \delta^n M_r$ and $M_r = \delta^n M_k$, where the base case is a valley such that $M_0 \rightarrow M_r \leftarrow M_k$ with $m_l = 0$.

The results of Lemma 3.7 and Corollary 3.8 can be unified as follows. The main theorem shows that every term in the reduction sequence $l$s of $M_0 = \delta^n M_k$ generates a common contractum: For every term $M$ in $l$s, there exists a natural number $n \leq \max\{l, r\}$ such that $M^{s*}$ is a common contractum of $M_0$ and $M_k$. Moreover, there exist a term $N$ in $l$s and a natural number $m \leq \min\{l, r\}$ such that $N^{m*}$ is a common contractum of all the terms.
in $ls$.

**Theorem 3.9** (Main theorem for $\beta$-equality ([6]).) Let $M_0 = \frac{a}{b} M_k$ with reduction sequence $[M_0, \ldots, M_k]$. Let $l = \sharp[l[0,k]$ and $r = \sharp[r[0,k]$.

Then there exist the following common reducts:

1. We have $M_0 \rightarrow M_{i+1}^{[r,i]}$ and $M_{i+1}^{[r,i]} \rightarrow M_k$ for each $i = 0, \ldots, r$. We also have $M_0 \rightarrow M_{r+1}^{[r,r+j]}$ and $M_{r+1}^{[r,r+j]} \rightarrow M_k$ for each $j = 0, \ldots, l$.

2. For every term $M$ in the reduction sequence, we have $M \rightarrow M_{m+1}^{[r]}$ where $m_l = \sharp[l[0,r]$.

**Proof.** Both 1 and 2 are proved similarly from Lemma 3.7, Corollary 3.8, and monotonicity. We show the case 2 here. Let $M_i$ be a term in the reduction sequence of $M_0 = \frac{a}{b} M_k$ where $0 \leq i \leq r$. Take $a = \sharp[r[0,i]$ then $M_n^{[0,0]}$ is a crossed point of $M_0 \rightarrow M_n^{[0]}$ and $M_i \rightarrow M_n^{[0,i]}$. From $M_i \rightarrow M_n^{[i]}$ and monotonicity, we have $M_i^{[i]} \rightarrow M_m^{[r,r+j]}$ where $m = \sharp[l[0,i] + \sharp[i,r]$. Hence, we have $M_i \rightarrow M_n^{[0,0]} \rightarrow M_i^{[r]} \rightarrow M_m^{[r,r+j]}$. The case of $r \leq i \leq k$ is also verified similarly.

Note that the case of $i = r$ and $j = l$ implies the main lemma, since $\sharp[r[0,k] = r$ and $\sharp[l[0,r + l] = \sharp[l[0,k] = l$. Note also that the case of $i = 0 = j$ implies the refinement, since $\sharp[l[0,r] = m_t = \sharp[r[0,k]$.

**Corollary 3.10** (Confluence). Let $P_n \rightarrow M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m (1 \leq n \leq m)$. Then we have $P_n \rightarrow Q_m^{[n]}$ and $Q_m \rightarrow Q_m^{[n]}$. We also have $P_n \rightarrow Q_m^{[n]}$ and $Q_m \rightarrow Q_m^{[n]}$.

**Proof.** From the main lemma and the refinement where $Q_0 \equiv M$.

4 Quantitative analysis of Church-Rosser theorem

Following the results and proof methods in the previous section, the size of common reducts and the number of reduction steps leading to a common reduct are investigated in detail in [6]. The method of a general principle and indeed can be extended to handle any system with the Z-property [5].

Let $(A, \rightarrow)$ be an abstract rewriting system where the reduction $\rightarrow$ is a binary relation on the set $A$. An element of $A$ is also called a term, and suppose that the size of a term $M$ is well defined, denoted by a natural number $|M|$.

Following Definitions 2.2 and 2.3, we define the reflexive transitive closure of $\rightarrow$ with a reduction sequence $ls$, denoted by $\rightarrow^n$ with length $n$ of $ls$. We also define the reflexive transitive symmetric closure of $\rightarrow$ with a sequence $ls$, denoted by $\rightarrow^n$ with length $n$ of $ls$. From the definition, $M = A N$ with sequence $ls$ if and only if there exists a finite sequence of terms $M_0, \ldots, M_n \in A (n \geq 0)$ such that $ls = [M_0, \ldots, M_n]$, $M_0 \equiv M$, $M_n \equiv N$ and either $M_i \rightarrow M_{i+1}$ or $M_i \leftarrow M_{i+1}$ for $0 \leq i \leq n - 1$. For natural numbers $i$ and $j$ with $0 \leq i \leq j \leq n$, we write $\sharp[i,j]$ for the number of occurrences of right arrow $\rightarrow$ appeared in $M_i = A M_j$, and $\sharp[i,j]$ for the number of occurrences of left arrow $\leftarrow$ appeared in $M_i = A M_j$.

For quantitative analysis, we prepare important measure functions, $\text{TermSize}$, $\text{Mon}$ and $\text{Rev}$.

**Definition 4.1** (TermSize). By induction on the derivation, we define $\text{TermSize}(M = A N)$ as follows:

1. If $M = A N$ with reduction sequence (list) $ls$, then $\text{TermSize}(M = A N)$ is defined by $\max(\text{map}(\text{fn}\ x \Rightarrow |x|)\ ls)$.

2. If $M = A N$ is derived from $N = A M$, then $\text{TermSize}(M = A N)$ is defined by $\text{TermSize}(N = A M)$.

3. If $M = A N$ is derived from $M = A P$ and $P = A N$, then define $\text{TermSize}(M = A N)$ as $\max\{\text{TermSize}(M = A P), \text{TermSize}(P = A N)\}$.

**Proposition 4.2** (TermSize). Let $M_0 = \frac{a}{b} M_k$ with sequence $ls$. For each term $M$ in $ls$, we have $|M| \leq \text{TermSize}(M_0 = \frac{a}{b} M_k)$.
Proof. By induction on the derivation of $\equiv_A$. \qed

We suppose an abstract rewriting system $(A, \to)$ having the following function $f$ from $A$ to $A$ together with measure functions (bound functions) $\text{Mon}$ and $\text{Rev}$ to the set of natural numbers, such that (i) if $M \to^n N$ ($n \geq 1$) then $f(M) \to^l f(N)$ where $l \leq \text{Mon}(M,n)$, and (ii) if $M \to N$ then $N \to^l f(M)$ where $l \leq \text{Rev}(|M|)$, provided that the measure functions are monotonic. We write $f^{n+1}(M) = f(f^n(M))$ and $f^0(M) = M$.

Then it is straightforward to reformulate Lemma 3.7 and Corollary 3.8 in terms of abstraction rewriting systems.

Proposition 4.3 (Lemma 3.7 revised). Let $M = \lambda_A N$ with length $k = l + r$, where $r = \sharp r[0,k]$, $l = \sharp l[0,k]$ and $B = \text{TermSize}(M = \lambda_A N)$. Then we have $f^l(M) \equiv^* N$ such that $a \leq \text{Main}(M = \lambda_A N)$, where the function $\text{Main}$ is defined by induction on $k$, as follows:
1. $\text{Main}(M \leftarrow N) = 1$
2. $\text{Main}(M \to N) = \text{Rev}(|M|)$
3. $\text{Main}(M = \lambda_A P \leftarrow Q) = \text{Main}(M = \lambda_A P) + 1$
4. $\text{Main}(M = \lambda_A P \to Q) = \text{Mon}(B,p) + \text{Rev}(B)$, where $p = \text{Main}(M = \lambda_A P)$.

Proof. From the proof of Lemma 3.7. Particularly in the last case where $f^{\sharp r[0,n] + 1}(M) \equiv^* f(P)$ $\equiv^b Q$, we have $a + b \leq \text{Mon}(|P|,p) + \text{Rev}(|P|) \leq \text{Mon}(B,p) + \text{Rev}(B)$.

\qed

Proposition 4.4 (Corollary 3.8 revised). Let $M = \lambda_A N$ with reduction sequence $[M_0, M_1, \ldots, M_k]$, where $r = \sharp r[0,k]$, $l = \sharp l[0,k]$ and $m_i = \sharp l[0,r]$. Then we have $M \equiv^a f^{m_1}(M_i) \equiv^b N$, where $a \leq \text{Main}(M_r = \lambda_A M)$ and $b \leq \text{Main}(M_r = \lambda_A N)$.

Proof. From Corollary 3.8 and Proposition 4.3. \qed

We remark that from Lemma 3.4 and Proposition 3.6, the measure function $\text{Main}$ is a function in the fourth level of the Grzegorczyk hierarchy in the case of $\lambda$-calculus [6].

5 Compositional Z-property

We begin with Dehornoy and van Oostrom’s Z theorem, and then extend it for compositional functions, called the compositional $Z$. It gives a sufficient condition for that a compositional function satisfies the $Z$-property, by dividing a rewriting system into two parts.

Definition 5.1 ((Weak) Z-property [15]). Let $(A, \to)$ be an abstract rewriting system, and $\to$ be the reflexive transitive closure of $\to$. Let $\to_x$ be another relation on $A$, and $\to_x$ be its reflexive transitive closure.

1. A mapping $f$ satisfies the weak $Z$-property for $\to$ by $\to_x$ if $M \to N$ implies $N \to_x f(M) \to_x f(N)$ for any $M \in A$.

2. A mapping $f$ satisfies the $Z$-property for $\to$ if it satisfies the weak $Z$-property for $\to$ by $\to$ itself.

When $f$ satisfies the (weak) $Z$-property, we also say that $f$ is (weakly) $Z$.

It becomes clear why we call it the $Z$-property when we draw the condition as the following diagram.

$\text{Main}(M) \equiv^a N$

Then $M \equiv^b f(M) \equiv^c f(N)$

Theorem 5.2 (Z theorem [5]). If there exists a mapping satisfying the $Z$-property for an abstract rewriting system, then it is confluent.

This theorem has been applied to confluence proofs for some variants of $\lambda$-calculus in [5],[13],[1] [16]. In fact, we can often prove that the usual complete developments have the $Z$-property.

The compositional $Z$ is the following, which is easily proved from Theorem 5.2 with the diagrams in Figure 1.

Theorem 5.3 (Compositional Z [15]). Let $(A, \to)$ be an abstract rewriting system, and $\to$ be $\to_1 \cup \to_2$. If there exist mappings $f_1, f_2 : A \to A$ such
that
(a) $f_1$ is Z for $\rightarrow_1$
(b) $M \rightarrow_1 N$ implies $f_2(M) \rightarrow f_2(N)$
(c) $M \rightarrow f_2(M)$ holds for any $M \in \text{Im}(f_1)$
(d) $f_2 \circ f_1$ is weakly Z for $\rightarrow_2$ by $\rightarrow$.

then $f_2 \circ f_1$ is Z for $(A, \rightarrow)$, and hence $(A, \rightarrow)$ is confluent.

One example of the compositional Z is a confluence proof for the $\beta\eta$-reduction on the untyped $\lambda$-calculus (although it can be directly proved by the Z theorem as in [13]). Let $\rightarrow_1 = \rightarrow_\eta$, $\rightarrow_2 = \rightarrow_\beta$, and $f_1$ and $f_2$ be the usual complete developments of $\eta$ and $\beta$, respectively. Then, it is easy to see the conditions of the compositional Z hold. The point is that we can forget the other reduction in the definition of each complete development.

Furthermore, we have another sufficient condition for the Z-property of compositional functions as follows. It is a special case of the compositional Z where $f_1(M) = f_1(M)$ holds for any $M \rightarrow_1 N$. All of the examples (except for $\beta\eta$ above) of the application of compositional Z in [15] are in this case.

**Corollary 5.4** ([15]). Let $(A, \rightarrow)$ be an abstract rewriting system, and $\rightarrow$ be $\rightarrow_1 \cup \rightarrow_2$. Suppose that there exist mappings $f_1, f_2 : A \rightarrow A$ such that
(a) $M \rightarrow_1 N$ implies $f_1(M) = f_1(N)$
(b) $M \rightarrow_1 f_1(M)$ for any $M$
(c) $M \rightarrow f_2(M)$ holds for any $M \in \text{Im}(f_1)$
(d) $f_2 \circ f_1$ is weakly Z for $\rightarrow_2$ by $\rightarrow$.

Then, $f_2 \circ f_1$ is Z for $(A, \rightarrow)$, and hence $(A, \rightarrow)$ is confluent.

**Proof.** It is easily proved from Theorem 5.3. The condition (a) in Theorem 5.3 comes from the new conditions (a) and (b), and (b) in Theorem 5.3 is not necessary since we have $f_2(f_1(M)) = f_2(f_1(N))$ for any $M \rightarrow_1 N$. □

Corollary 5.4 can be seen as generalization of the Z-property modulo, proposed by Accattoli and Kesner [1]. For an abstract rewriting system $(A, \rightarrow)$ and an equivalence relation $\sim$ on $A$, the reduction modulo $\sim$, denoted $M \rightarrow_\sim N$, is defined as $M \sim P \rightarrow Q \sim N$ for some $P$ and $Q$. The Z-property modulo says that it is a sufficient condition for the confluence of $\rightarrow_\sim$ that there exists a mapping which is well-defined on $\sim$ and weakly Z for $\rightarrow$ by $\rightarrow_\sim$. If we consider $\sim$ as the first reduction relation $\rightarrow_1$, and define $f_1(M)$ as a fixed representative of the equivalence class including $M$, then the conditions of the Z-property modulo implies the conditions of the compositional Z, since the reflexive transitive closure of $\rightarrow_1 \cup \sim$ is $\rightarrow_\sim$.

### 6 Quantitative analysis under compositional Z-property

The two approaches in Sections 4 and 5 are naturally unified into a single framework. For this, we introduce the compositional Z-property together with measure functions $\text{Mon}$, $\text{Rev}$ and $\text{Eval}$ as modules of bound functions.

**Proposition 6.1.** Let $(A, \rightarrow)$ be an abstract rewriting system, and $\rightarrow$ be $\rightarrow_1 \cup \rightarrow_2$. Suppose that there exist functions $f_1, f_2 : A \rightarrow A$ and mono-
tomic measure functions $\text{Rev}_1$, $\text{Rev}_2$, $\text{Eval}_2$ and $\text{Mon}$ such that all of the following conditions hold.

1. $f_1$ is $Z$ for $\rightarrow_1$:
   
   \[ M \rightarrow_1 N \text{ then } N \rightarrow a^l f_1(M) \rightarrow_1 f_1(N), \]
   
   where $a \leq \text{Rev}_1(|M|)$.

2. $f_2 \circ f_1$ is weakly $Z$ for $\rightarrow_2$ by $\rightarrow$:
   
   \[ M \rightarrow_2 N \text{ then } N \rightarrow a^l f_2(f_1(M)) \rightarrow f_2(f_1(N)), \]
   
   where $a \leq \text{Rev}_2(|M|)$.

3. $M \rightarrow a^l f_3(M)$ holds for any $M \in \text{Im}(f_1)$, where $a \leq \text{Eval}_2(|M|)$.

4. $f_2 \circ f_1$ is weakly $Z$ for $\rightarrow_2$ by $\rightarrow$:
   
   \[ M \rightarrow_2 N \text{ then } N \rightarrow a^l f_2(f_1(M)) \rightarrow f_2(f_1(N)), \]
   
   where $a \leq \text{Rev}_2(|M|)$.

5. $M \rightarrow a^l N$ then $f_3(f_1(M)) \rightarrow b f_2(f_1(N))$, where $b \leq \text{Mon}(|M|, a)$.

6. $M =_A^k N$ then $M \rightarrow^* N$ if and only if $f = f_2 \circ f_1$ is $Z$ for $\rightarrow_1$.

Proof. From Proposition 6.1.

\[ \square \]

7 Concluding remarks

In this paper, first we investigated directly the Church-Rosser theorem for $\beta$-equality constructively from the viewpoint of Takahashi translation [22]. Our investigation shows that a common contractum of $M$ and $N$ such that $M =_\beta N$ is determined by (i) $M$ and the number of occurrences of reduction ($\rightarrow$) appeared in $=_\beta$, and also by (ii) $N$ and that of reversed reduction ($\leftarrow$). In terms of iteration of Takahashi translation, this characterization of the Church-Rosser theorem makes it possible to analyse how large common contractums are and how many reduction-steps are required to obtain them. From this, we obtained an upper bound function for the theorem in the fourth level of the Grzegorczyk hierarchy.

Next, we demonstrated that the proof method is available as well under a general framework of the compositional $Z$ [15]. For this extension, the measure functions constructed for quantitative analysis of the Church-Rosser theorem are naturally abstracted as fundamental modules of bound functions. This approach makes it possible to analyze quantitative properties of abstract rewriting systems under the framework of the compositional $Z$.

Corollary 5.4 can be seen as generalization of the $Z$-property modulo, proposed by [1]. Moreover, it would be interesting to extend the compositional $Z$-property to cooperate with confluent modulo equivalence such as in [10] for applications to practical problems.

References


